Μια εισαγωγή στα Μαθηματικά για Οικονομολόγους

Έστω δυο προτάσεις A και B

«A είναι <u>αναγκαία συνθήκη</u> για την B»

"A is necessary for B"

$A \Leftarrow B$

«A είναι <u>ικανή συνθήκη</u> για την B»

"A is sufficient for B"

$A \Rightarrow B$

«A είναι <u>ικανή και αναγκαία συνθήκη</u> για την B»

"A if-f B" or "A is necessary and sufficient for B"

$A \Leftrightarrow B$

Τα θεωρήματα έχουν συνήθως τη μορφή

$$A \Rightarrow B$$

A = προϋπόθεση
B = συμπέρασμα

Για να αποδείξουμε το θεώρημα θα πρέπει να δείξουμε την ορθότητα του συμπεράσματος, B , χρησιμοποιώντας την αλήθεια της προϋπόθεσης, A, με όρους βασικής λογικής

→ constructive proof: $A \Rightarrow B$

→ contrapositive proof: $\sim A \Leftarrow \sim B$

→ proof by contradiction: Υποθέσεις: A αληθές και B όχι αληθές τότε να καταλήξουμε σε

$\Rightarrow$ ΑΤΟΠΟ

→ Ένα σύνολο S είναι υποσύνολο (subset) ενός άλλου συνόλου T :

$$S \subset T \Leftrightarrow x \in S \Rightarrow x \in T$$

→ Δυο σύνολα είναι ισα, εάν περιέχουν ακριβώς τα ίδια στοιχεία:

$$S = T \Leftrightarrow x \in S \Rightarrow x \in T \quad \text{and} \quad x \in T \Rightarrow x \in S$$

$$\Leftrightarrow \qquad S \subset T \qquad \text{and} \qquad T \subset S$$

→ Κενό σύνολο

→ Συμπλήρωμα (complement) συνόλου

→ Ένωση (union) συνόλων $S \cup T \Leftrightarrow \{x \setminus x \in S \quad or \quad x \in T\}$

→ Τομή (intersection) συνόλων $S \cap T \Leftrightarrow \{x \setminus x \in S \quad \text{and} \quad x \in T\}$

→ Γινόμενο (product of two sets) συνόλων: $S \times T \Leftrightarrow \{(s,t) \setminus s \in S,\ t \in T\}$

→ Στη Μικροοικονομική Θεωρία περιοριζόμαστε στα υποσύνολα:

$$\mathbb{R}^n_{++} \subset \mathbb{R}^n_{+} \subset \mathbb{R}^n$$

→ Θα γράφω

$$x \geq 0 \quad \text{ή} \quad x \in \mathbb{R}^n_{+} \quad \text{και θα εννοώ} \quad x_i \geq 0, \quad i = 1, 2..n$$

$$x >> 0 \quad \text{ή} \quad x \in \mathbb{R}^n_{++} \quad \text{και θα εννοώ} \quad x_i > 0, \quad i = 1, 2..n$$

Κυρτά σύνολα (convex sets) στον $\mathbb{R}^n$

$S \subset \mathbb{R}^n$ είναι κυρτό σύνολο εάν για κάθε $x^1 \in S$ και $x^2 \in S$

$$tx^1 + (1-t)x^2 \in S, \qquad \forall t \in [0,1]$$

$$f : A \to B$$

A = domain B = range

$$f(a) = b$$

a $\in A$ = όρισμα της f, b $\in B$ = τιμή, εικόνα της f στο α

Μια συνάρτηση $f: A \to B$ λέγεται **1-1 ή αμφιμονοσήμαντη**, όταν αντιστοιχίζει κάθε όρισμα σε αποκλειστικά δική του τιμή ή αλλιώς, όταν διαφορετικά ορίσματα απεικονίζονται σε διαφορετικές τιμές:

$$\text{αν } a \neq a' \text{ τότε } f(a) \neq f(a')$$

Μια συνάρτηση $f: A \to B$ λέγεται **επί** (onto), όταν δεν υπάρχει στοιχείο στο B που να μην είναι η εικόνα κάποιου στοιχείου του A:

$$\forall b \in B, \quad \exists a \in A: \quad b = f(a)$$

Μια συνάρτηση είναι **αντιστρέψιμη** ($\exists f^{-1}$) αν είναι 1-1 και επί:

$$f^{-1}: B \to A \qquad f^{-1}(b) = a \Leftrightarrow b = f(a)$$

Μετρική είναι απλά ένα μέγεθος μέτρησης της απόστασης

Μετρικός χώρος είναι ένα σύνολο στο οποίο έχει οριστεί μια έννοια απόστασης (μια μετρική).

Μετρικοί χώροι	Μετρική
$\mathbb{R}$	$d(x^1, x^2) = \left\| x^1 - x^2 \right\|$
$\mathbb{R}^2 = \{(x_1, x_2) \setminus x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$	$d(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2}$
$\mathbb{R}^n = \{(x_1, x_2, ..x_n) \setminus x_i \in \mathbb{R}, i = 1, 2...n\}$	$d(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + ... + (x_n^2 - x_n^1)^2}$

Definition: Open and Closed ε-balls

1. The **open ε-ball** with center x^0 and radius $\varepsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$:

$$B_\varepsilon(x^0) \equiv \left\{ x \in \mathbb{R}^n \setminus d(x^0, x) < \varepsilon \right\}$$

2. The **closed ε-ball** with center x^0 and radius $\varepsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$:

$$B_\varepsilon^*(x^0) \equiv \left\{ x \in \mathbb{R}^n \setminus d(x^0, x) \leq \varepsilon \right\}$$

Definition: Open sets in $\mathbb{R}^n$

$S \subset \mathbb{R}^n$ is an **open set** if, for all $x \in S$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$

Theorem: On Open Sets in $\mathbb{R}^n$

1. $\varnothing$ is an open set
2. $\mathbb{R}^n$ is an open set
3. The union of open sets is an open set
4. The intersection of any finite collection of open sets is an open set

<u>Definition</u>: Closed sets in $\mathbb{R}^n$

$S \subset \mathbb{R}^n$ is a **closed set** if, its complement S^c, is an open set

<u>Theorem</u>: On Closed Sets in $\mathbb{R}^n$

1. $\varnothing$ is a closed set

2. $\mathbb{R}^n$ is a closed set

3. The union of any finite collection of closed sets is a closed set

4. The intersection of closed sets is a closed set

<u>Definition</u>: Bounded sets in $\mathbb{R}^n$

A set $S \subset \mathbb{R}^n$ is called **bounded** if it is entirely contained within some ε-ball (either open or closed). That is S is bounded if there exists some $\varepsilon > 0$ such that $S \subset B_\varepsilon(x)$ for some $x \in \mathbb{R}^n$.

Definition (Heine – Borel): Compact sets in $\mathbb{R}^n$

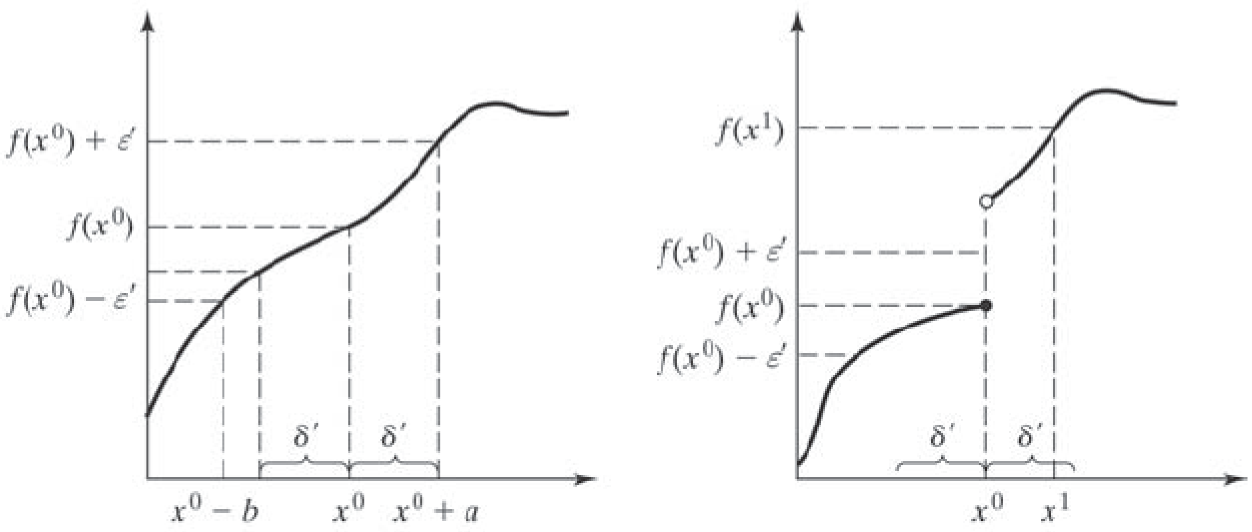
A set $S \subset \mathbb{R}^n$ is called **compact** if it is closed and bounded

Definition: Caushy continuity

Let $D \subset \mathbb{R}^m$ and let $f : D \to \mathbb{R}^n$. The function f is **continuous** at the point $x^0 \in D$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$f\left(B_\delta(x^0) \cap D\right) \subset B_\varepsilon\left(f(x^0)\right)$$

If f is continuous at every point $x \in D$, then it is called a continuous function.



(a) (b)

Theorem: Continuity and inverse images

Let $D \subset \mathbb{R}^m$. The following conditions are equivalent:

1. $f : D \to \mathbb{R}^n$ is continuous
2. For every open ball B in $\mathbb{R}^n$, $f^{-1}(B)$ is open in D
3. For every open $S \subset \mathbb{R}^n$, $f^{-1}(S)$ is open in D

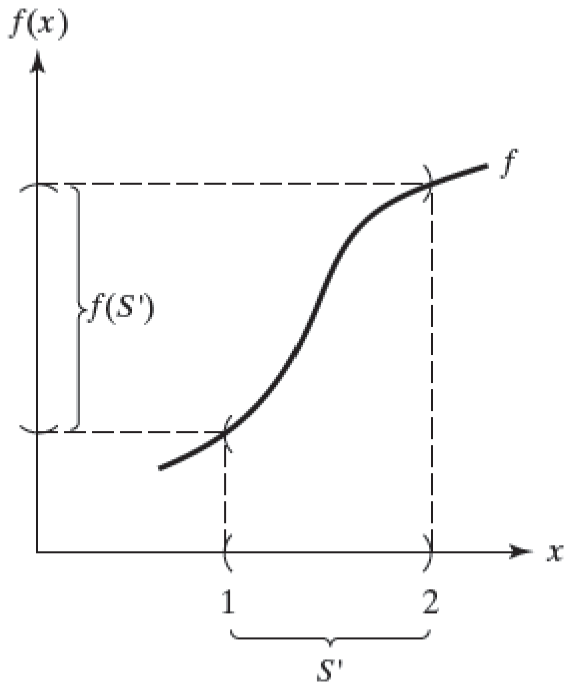
Theorem: The Continuous Image of a Compact Set is a Compact Set

Let $D \subset \mathbb{R}^m$ and let $f : D \to \mathbb{R}^n$ be a continuous function. If $S \subset D$ is compact (closed and bounded) then its image $f(S) \subset \mathbb{R}^n$ is a compact set.

Theorem (Weierstrass): Existence of Extreme Values

Let $f : S \to \mathbb{R}$ be a **continuous real-valued function**, where S is a **nonempty compact** subset of $\mathbb{R}^n$. Then there exists a vector $x^* \in S$ and a vector $\overline{x} \in S$ such that:

$$f(x^*) \leq f(x) \leq f(\overline{x}), \qquad \forall x \in S$$



<u>Definition</u>: Real Valued Functions

$f : D \to R$ is a **real-valued function** if D is any set and $R \subset \mathbb{R}$

<u>Definition</u>: Increasing Real Valued Functions

Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$. Then

f is **increasing** if $f(x^0) \geq f(x^1)$ whenever $x^0 \geq x^1$

f is **strictly increasing** if $f(x^0) > f(x^1)$ whenever $x^0 >> x^1$

f is **strongly increasing** if $f(x^0) > f(x^1)$ whenever $x^0 \geq x^1$ and $x^0 \neq x^1$

Definition: Decreasing Real Valued Functions

Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$. Then

f is **decreasing** if $f(x^0) \leq f(x^1)$ whenever $x^0 \geq x^1$

f is **strictly decreasing** if $f(x^0) < f(x^1)$ whenever $x^0 >> x^1$

f is **strongly decreasing** if $f(x^0) < f(x^1)$ whenever $x^0 \geq x^1$ and $x^0 \neq x^1$

<u>Definition</u>: Level Sets

$L(y^0)$ is a **level set** of the real-valued function $f : D \to R$ if-f

$$L(y^0) = \left\{ x \setminus x \in D,\, f(x) = y^0 \right\}, \text{ where } y^0 \in R \subset \mathbb{R}$$

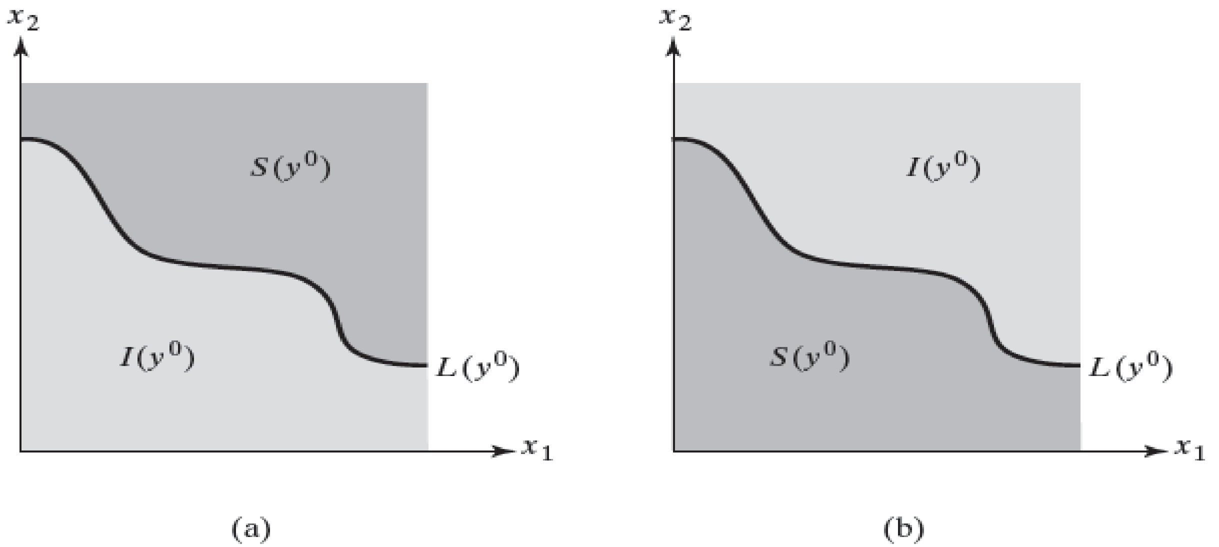
<u>Definition</u>: Superior and Inferior Sets

1. $S(y^0) \equiv \left\{ x \setminus x \in D,\, f(x) \geq y^0 \right\}$ is called the **superior set** for level y^0

2. $I(y^0) \equiv \left\{ x \setminus x \in D,\, f(x) \leq y^0 \right\}$ is called the **inferior set** for level y^0

3. $S'(y^0) \equiv \left\{ x \setminus x \in D,\, f(x) > y^0 \right\}$ is called the **strictly superior set** for level y^0

4. $I'(y^0) \equiv \left\{ x \setminus x \in D,\, f(x) < y^0 \right\}$ is called the **strictly inferior set** for level y^0



Definition: Superior and Inferior Sets

1. $S(y^0) \equiv \{x \setminus x \in D, f(x) \geq y^0\}$ is called the **superior set** for level y^0

2. $I(y^0) \equiv \{x \setminus x \in D, f(x) \leq y^0\}$ is called the **inferior set** for level y^0

3. $S'(y^0) \equiv \{x \setminus x \in D, f(x) > y^0\}$ is called the **strictly superior set** for level y^0

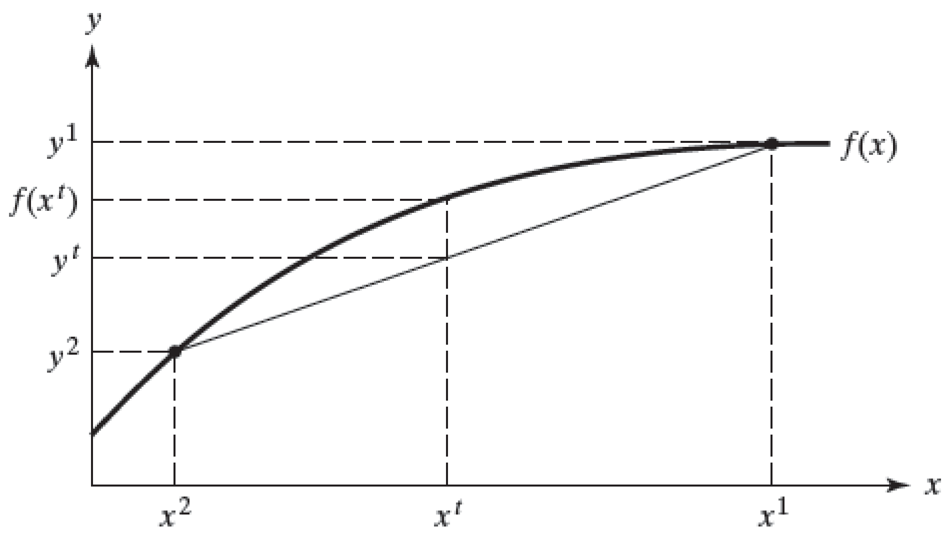
4. $I'(y^0) \equiv \{x \setminus x \in D, f(x) < y^0\}$ is called the **strictly inferior set** for level y^0

Definition: Concave Function

$f: D \to R$ is a **concave function** if for all $x^1, x^2 \in D$

$$f(\overline{x}) \geq tf(x^1) + (1-t)f(x^2) \qquad \forall t \in [0,1]$$

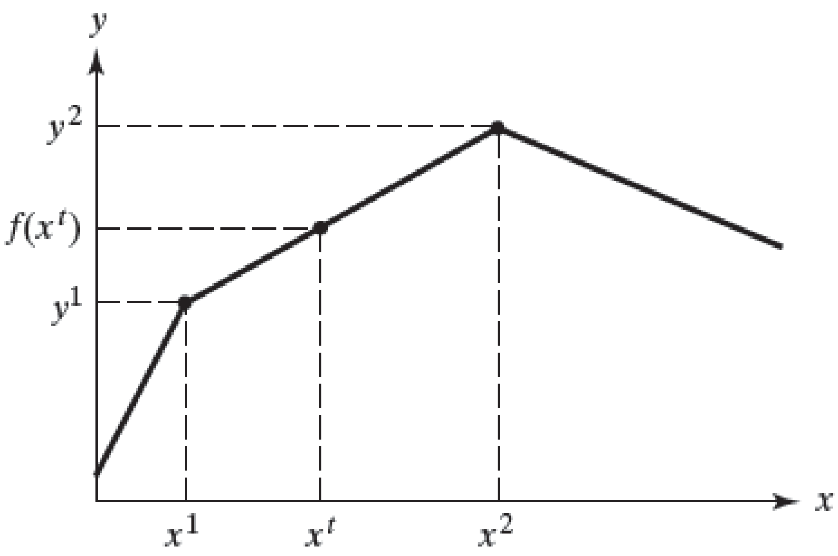
where $\overline{x} = tx^1 + (1-t)x^2$ denotes the convex combination of x^1, x^2



Theorem: Points on and below the Graph of a Concave function always form a Convex Set

Let $A \equiv \{(x, y) \setminus x \in D, f(x) \geq y\}$ be the set of points "on and below" the graph of $f : D \to R$, where $D \subset \mathbb{R}^n$ is a convex set and $R \subset \mathbb{R}$. Then

f is concave function $\Leftrightarrow$ A is a convex set



<u>Definition</u>: Strictly Concave Functions

$f : D \to R$ is a **strictly concave function** if-f

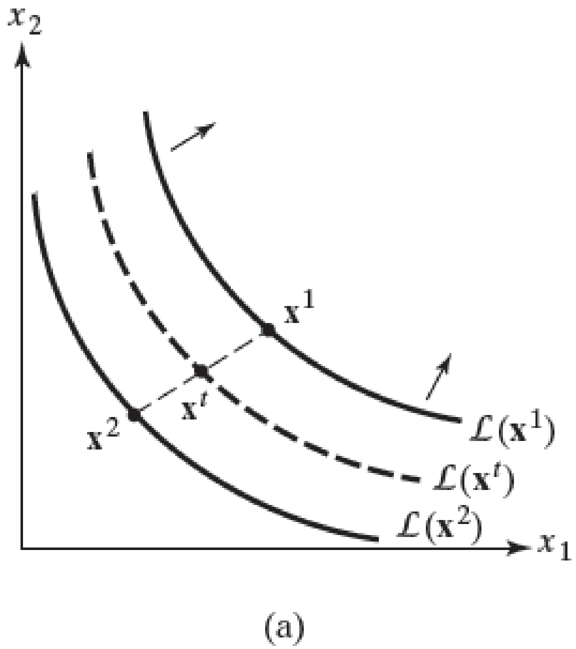
for all $x^1 \neq x^2$ in D: $f\left(tx^1 + (1-t)x^2\right) > tf(x^1) + (1-t)f(x^2) \quad \forall t \in (0,1)$

Quasi-Concave Functions

Definition: Quasi-Concave Function

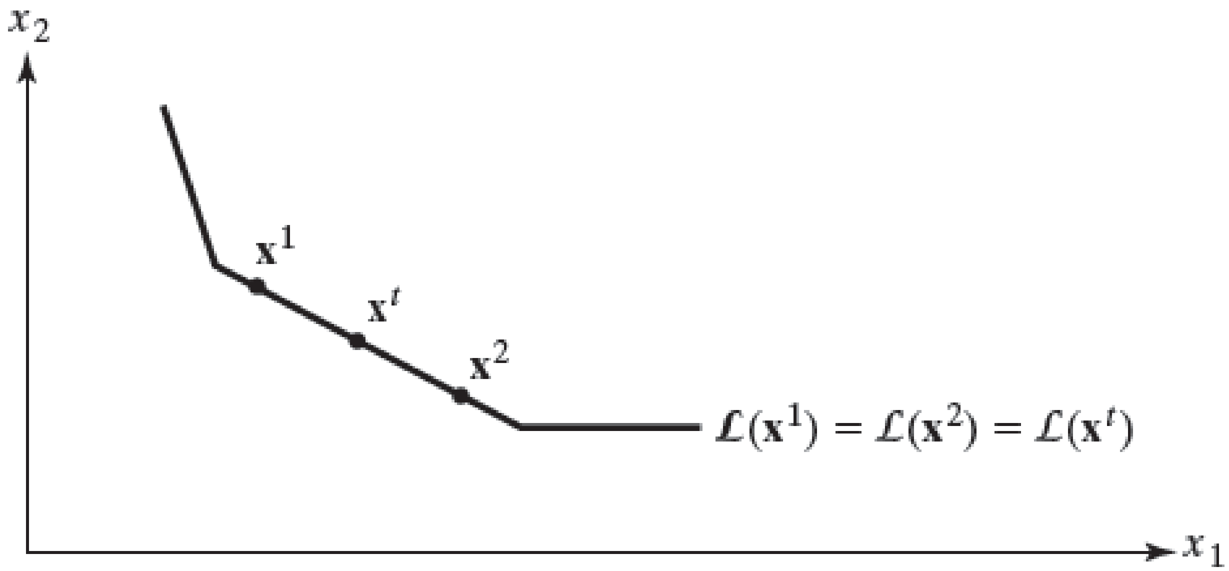
$f : D \rightarrow R$ is **quasi-concave** if-f for all $x^1, x^2 \in D$:

$$f\left(tx^1 + (1-t)x^2\right) \geq \min\left[\, f(x^1),\, f(x^2) \,\right] \qquad \forall t \in [0,1]$$



<u>Theorem</u>: Quasi-Concavity and the Superior Sets

$f: D \to \mathbb{R}$ is a quasi-concave function if-f $S(y)$ is a convex set for all $y \in \mathbb{R}$



Quasi-Concave Functions

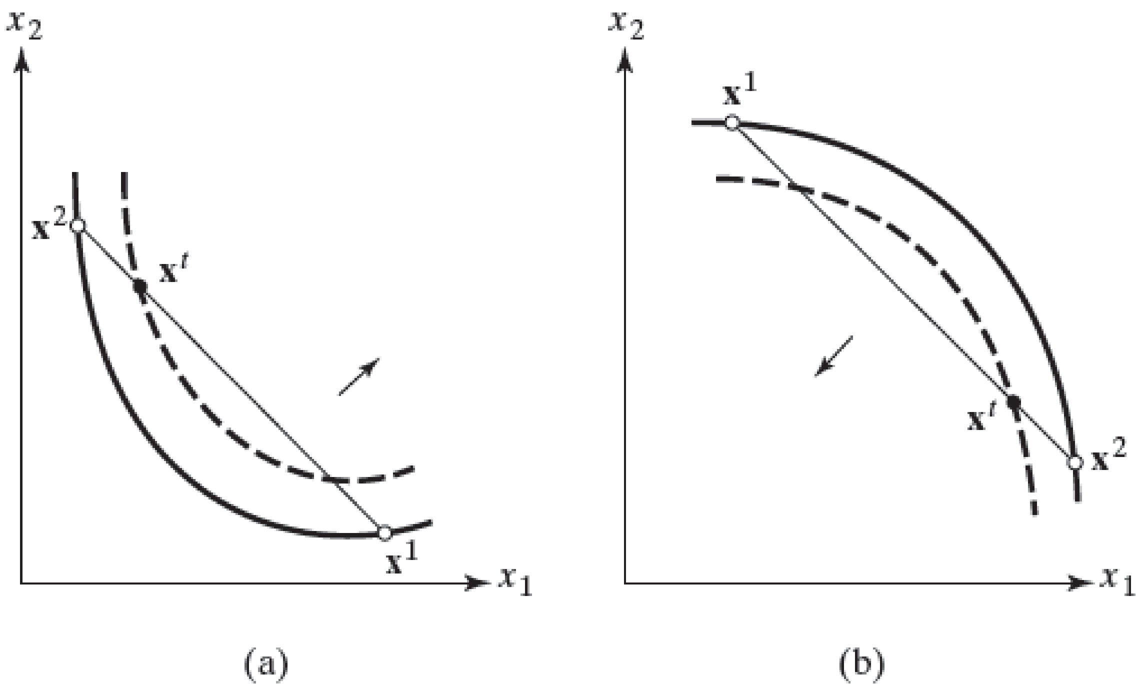
Definition: Strictly Quasi-Concave Function

$f: D \to R$ is **strictly quasi-concave** if-f for all $x^1 \neq x^2 \in D$:

$$f\left(tx^1 + (1-t)x^2\right) > \min\left[\ f(x^1),\ f(x^2)\ \right] \qquad \forall t \in (0,1)$$

Theorem: Strictly Quasi-Concavity and the Superior Sets

$f: D \to \mathbb{R}$ is a strictly quasi-concave function if-f $S(y)$ is a strictly convex set for all $y \in \mathbb{R}$


x_2
$\mathbf{x}^2$
$\mathbf{x}^t$
$\mathbf{x}^1$
x_1
(a)
x_2
$\mathbf{x}^1$
$\mathbf{x}^t$
$\mathbf{x}^2$
x_1
(b)

<u>Theorem</u>: Concavity implies Quasi-concavity

A (strictly) concave function is always (strictly) quasi-concave

<u>Theorem</u>: Cobb-Douglas Function

Every Cobb-Douglas function $f(x_1, x_2) = Ax_1^a x_2^b$ with $A, a, b > 0$ is quasi-concave.

Definition: Convex and Strictly Convex Functions

1. $f : D \rightarrow R$ is a **convex function** if for all $x^1, x^2 \in D$

$$f\left(tx^1 + (1-t)x^2\right) \leq tf(x^1) + (1-t)f(x^2) \qquad \forall t \in [0,1]$$

2. $f : D \rightarrow R$ is a **strictly convex function** if for all $x^1 \neq x^2 \in D$

$$f\left(tx^1 + (1-t)x^2\right) < tf(x^1) + (1-t)f(x^2) \qquad \forall t \in (0,1)$$

<u>Theorem</u>: Points on and Above the Graph of a Concave function always form a Convex Set

Let $A^* \equiv \{(x,y) \setminus x \in D, f(x) \leq y\}$ be the set of points "on and above" the graph of $f: D \to R$, where $D \subset \mathbb{R}^n$ is a convex set and $R \subset \mathbb{R}$. Then

$$f \text{ is convex function} \Leftrightarrow A^* \text{ is a convex set}$$

<u>Definition</u>: Quasi-Convex and Strictly q-convex Function

1. $f: D \to R$ is **quasi-convex** if-f for all $x^1, x^2 \in D$:

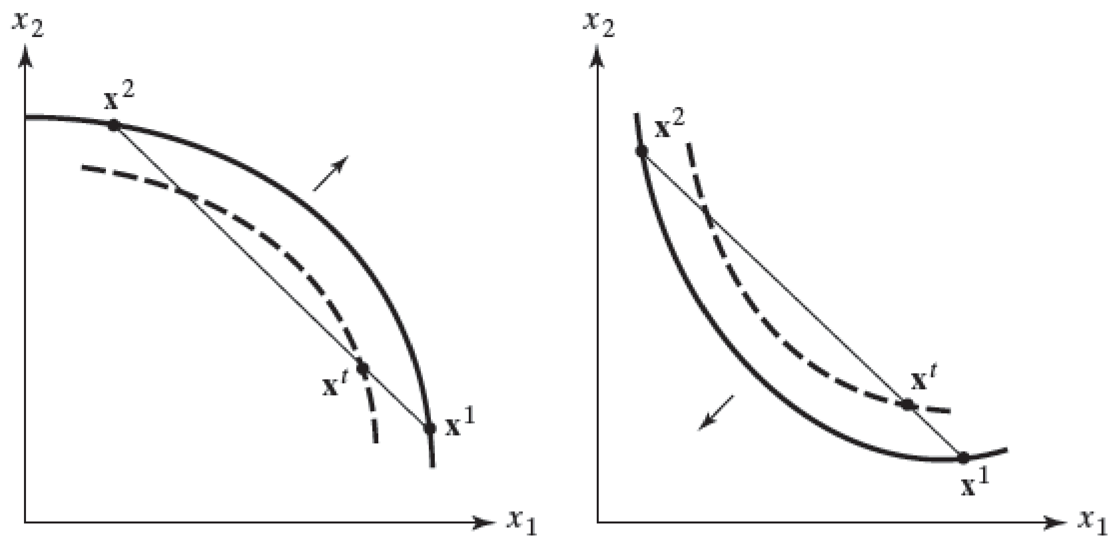
$$f\left(tx^1 + (1-t)x^2\right) \leq \max\left[\ f(x^1), f(x^2)\ \right] \qquad \forall t \in [0,1]$$

2. $f: D \to R$ is **strictly quasi-convex** if-f for all $x^1 \neq x^2 \in D$:

$$f\left(tx^1 + (1-t)x^2\right) < \max\left[\ f(x^1), f(x^2)\ \right] \qquad \forall t \in (0,1)$$

Theorem: (Strictly) Quasi-Convex and the Inferior Sets

$f : D \to \mathbb{R}$ is a (strictly) quasi-convex function if-f $I(y)$ is a (strictly) convex set for all $y \in \mathbb{R}$



(a) (b)

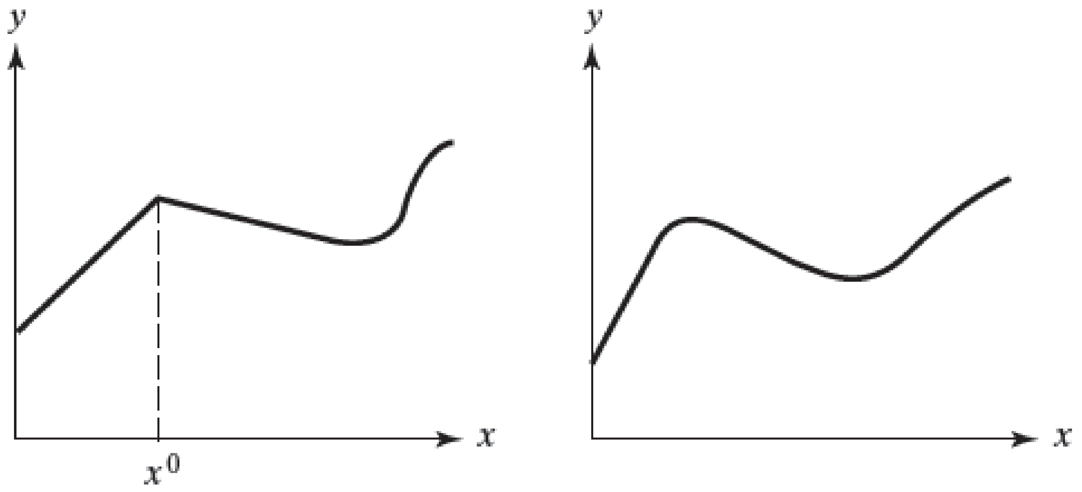
Theorem: (Strictly) Quasi-Convex and (Strictly) Quasi-Concave functions

$f(x)$ is a (strictly) quasi-concave function if-f $-f(x)$ is a (strictly) quasi-convex function

f is concave	$\Longleftrightarrow$	the set of points *beneath* the graph is convex
f is convex	$\Longleftrightarrow$	the set of points *above* the graph is convex
f quasiconcave	$\Longleftrightarrow$	superior sets are convex sets
f quasiconvex	$\Longleftrightarrow$	inferior sets are convex sets
f concave	$\Rightarrow$	f quasiconcave
f convex	$\Rightarrow$	f quasiconvex
f (strictly) concave	$\Longleftrightarrow$	$-f$ (strictly) convex
f (strictly) quasiconcave	$\Longleftrightarrow$	$-f$ (strictly) quasiconvex

Calculus and Optimization

Functions of a single variable



For Constants, α : $$\frac{d}{dx}(\alpha)=0.$$

For Sums: $$\frac{d}{dx}[f(x) \pm g(x)]=f'(x) \pm g'(x).$$

Power Rule: $$\frac{d}{dx}(\alpha x^n)=n\alpha x^{n-1}.$$

Product Rule: $$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x) + f'(x)g(x).$$

Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Chain Rule: $$\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x).$$

Calculus and Optimization

Functions of a single variable

Theorem:

Suppose $f : D \to R$, $D \subset \mathbb{R}$, $R \subset \mathbb{R}$ is twice continuously differentiable

1. f is concave $\Leftrightarrow f''(x) \le 0, \quad \forall x \in D$

2. f is convex $\Leftrightarrow f''(x) \ge 0, \quad \forall x \in D$

Moreover,

1. if $f''(x) < 0, \quad \forall x \in D$ then f is strictly concave

2. if $f''(x) > 0, \quad \forall x \in D$ then f is strictly convex

Calculus and Optimization

Functions of several variables

Calculus and Optimization

Functions of several variables

Theorem: Young's Theorem

For any twice continuously differentiable function $f(x), x \in \mathbb{R}^n$:

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}, \qquad \forall i, j$$

(Leading) Principal Minors of a Matrix

Definition: Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A formed by deleting $n-k$ columns and the same $n-k$ rows from A is called a kth order principal submatrix of A. The determinant of that principal submatrix is called a k **th order principal minor of** A.

Definition: Let A be an $n \times n$ matrix. The kth order principal submatrix of A obtained by deleting *the last* $n-k$ rows and *the last* $n-k$ columns from A is called the kth order *leading* principal submatrix of A. Its determinant is called the k **th order *leading* principal minor of** A

Definiteness of a Matrix

Theorem: Definiteness of a matrix

Let A be an $n \times n$ symmetric matrix

(a) A is **positive definite** if-f all its leading principal minors are **strictly positive**

(b) A is **negative definite** if-f its leading principal minors **alternate in sign** as follows:

$$|A_1| < 0, |A_2| > 0, |A_3| < 0.........$$

Semi-Definiteness of a Matrix

Theorem: Semi-Definiteness of a matrix

Let A be an $n \times n$ symmetric matrix

(a) A is **positive semi-definite** if-f every principal minor is ≥ 0

(b) A is **negative semi-definite** if-f every principal minor of odd order ≤ 0

and every principal minor of even order ≥ 0

Border Matrices

$$\bar{H} = \begin{bmatrix} 0 & f_1 & f_2 & \cdots f_n \\ f_1 & f_{11} & f_{12} & \cdots f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots f_{2n} \\ \vdots & & & \\ f_n & f_{n1} & f_{n2} & \cdots f_{nn} \end{bmatrix}$$

Border Matrices

<u>Theorem</u>: Definiteness of a bordered matrix

Let $\bar{H}$ be a symmetric bordered matrix

(a) $\bar{H}$ is **positive definite** if-f all its bordered principal minors are **strictly negative**

i.e

$$|\bar{H}_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} < 0 \qquad |\bar{H}_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} < 0 \ldots\ldots |\bar{H}| < 0$$

(b) $\bar{H}$ is **negative definite** if-f its bordered principal minors **alternate in sign** as follows:

$$|\bar{H}_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} < 0 \qquad |\bar{H}_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} > 0 \qquad |\bar{H}_3| < 0 \quad \ldots.$$

Border Matrices

Theorem: Semi-Definiteness of a bordered matrix

Let $\bar{H}$ be a symmetric bordered matrix

(a) $\bar{H}$ is **positive semi-definite** if-f every bordered principal minor is ≤ 0

(b) $\bar{H}$ is **negative semi-definite** if-f every bordered principal minor of odd order ≤ 0

and every bordered principal minor of even order ≥ 0

(Border) Matrices and (Quasi) Concavity/Convexity

Theorem: Concavity – Convexity in Many Variables

Let D be a convex subset of $\mathbb{R}^n$ on which f is twice continuously differentiable

f is concave (convex) $\Leftrightarrow$ $H(x)$ is negative (positive) semi-definite, $\forall x \in D$

Moreover

If $H(x)$ is negative (positive) definite $\forall x \in D$ then f is strictly concave (convex)

<u>Theorem</u>: Concavity – Convexity and Second-Order Own Partial Derivatives

Let $f : D \to R$ be a twice continuously differentiable function

1. If f concave $\Rightarrow$ $f_{ii}(x) \leq 0 \qquad i = 1, 2, \ldots n \qquad \forall x$

2. If f convex $\Rightarrow$ $f_{ii}(x) \geq 0 \qquad i = 1, 2, \ldots n \qquad \forall x$

(Border) Matrices and (Quasi) Concavity/Convexity

Theorem: Quasi-Concavity (Convexity) in many variables

Let D be a convex subset of $\mathbb{R}^n$ on which f is twice continuously differentiable

f is **quasi** concave (convex) $\Leftrightarrow$ $\bar{H}(x)$ is negative (positive) semi-definite, $\forall x \in D$

Moreover

If $\bar{H}(x)$ is negative (positive) definite $\forall x \in D$ then f is strictly quasi -concave (quasi-convex)

Homogeneous Functions

Definition: Homogeneous Functions

A real-valued function $f(x)$ is called homogeneous of degree k, if

$$f(tx) = t^k f(x) \qquad \forall t > 0$$

Homogeneous Functions

Theorem: Partial Derivatives of Homogeneous Functions

If $f(x)$ is h.o.d. k, its partial derivatives are h.o.d. k-1

Theorem: Euler's Theorem

Let $f(x)$ be a continuously differentiable **homogeneous function of degree k** on $\mathbb{R}_+^n$

Then for all x

$$x_1 \frac{\partial f(x)}{\partial x_1} + x_2 \frac{\partial f(x)}{\partial x_2} + \ldots + x_n \frac{\partial f(x)}{\partial x_n} = kf(x)$$

Optimization

Maxima and minima for single-variable functions

Consider the function of a single-variable $f(x) = y$ and assume it is differentiable

when we say the function achieves a **local maximum** at x^*, we mean that

$$f(x^*) \geq f(x), \quad \forall x \in B_\varepsilon(x^*)$$

when we say the function achieves a **global maximum** at x^*, we mean that

$$f(x^*) \geq f(x), \quad \forall x \in D$$

unique local maximum at x^* if $f(x^*) > f(x), \quad \forall x \neq x^* \in B_\varepsilon(x^*)$

unique global maximum at x^* if $f(x^*) > f(x), \quad \forall x \neq x^* \in D$

Optimization

Maxima and minima for single-variable functions

Theorem:

(a) If $f'(x_0) = 0$ and $f''(x_0) < 0$ then x_0 is **local max** of f

(b) If $f'(x_0) = 0$ and $f''(x_0) > 0$ then x_0 is **local min** of f

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$ then x_0 can be max, min, or neither

Optimization

Maxima and minima for single-variable functions

If $f(x)$ is a twice continuously differentiable function whose domain is an interval I, then

(a) If $f'(x_0)=0$ and $f''(x)<0, \forall x \in I$ then x_0 is a **global max** of f

(b) If $f'(x_0)=0$ and $f''(x)>0, \forall x \in I$ then x_0 is a **global min** of f

Optimization
Real-valued functions of n-variables

Definition:

Let $f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$

(1) A point $x^* \in D$ is a **global max** if $f(x^*) \geq f(x), \quad \forall x \in D$

(1*) A point $x^* \in D$ is a **unique global max** if $f(x^*) > f(x), \ \forall x \in D$ and $x^* \neq x$

(2) A point $x^* \in D$ is a **local max** if $f(x^*) \geq f(x), \quad \forall x \in B_\varepsilon(x^*) \cap D$

(2*) A point $x^* \in D$ is a **unique local max** if

$$f(x^*) > f(x), \quad \forall x \in B_\varepsilon(x^*) \cap D \text{ and } x^* \neq x$$

Optimization

Real-valued functions of n-variables

Theorem:

Let $f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$ be a twice continuously differentiable function

If $x^* \in D$ is a local max or min of f and if x^* is an interior point of D, then x^* solves the system

$$\frac{\partial f(x^*)}{\partial x_1} = 0$$

$$\frac{\partial f(x^*)}{\partial x_2} = 0$$

$$\ldots$$

$$\frac{\partial f(x^*)}{\partial x_n} = 0$$

Optimization

Real-valued functions of n-variables

SECOND ORDER CONDITIONS

Theorem: Sufficient Conditions

Let $f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$ be a twice continuously differentiable function

Suppose that x^* satisfies $\frac{\partial f(x^*)}{\partial x_i} = 0, \quad i = 1, 2, \ldots n$ and that the leading principal minors of $H(x^*)$ alternate in sign

$$|H_1| < 0, |H_2| > 0, |H_3| < 0 \ldots\ldots\ldots$$

at x^*. Then x^* is a **unique local max** of f

Optimization

Real-valued functions of n-variables

SECOND ORDER CONDITIONS

Theorem: Necessary Conditions

Let $f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$ be a twice continuously differentiable function

(1) If $f(x)$ reaches a local interior maximum at x^* then $\dfrac{\partial f(x^*)}{\partial x_i} = 0, \quad i = 1, 2, \ldots n$

and $H(x^*)$ is negative semi-definite

(2) If $f(x)$ reaches a local interior minimum at $\tilde{x}$ then $\dfrac{\partial f(\tilde{x})}{\partial x_i} = 0, \quad i = 1, 2, \ldots n$

and $H(\tilde{x})$ is positive semi-definite

Optimization

Real-valued functions of n-variables

Theorem: Global Theorem

Let $f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$ be a twice continuously differentiable function which is [strictly] CONCAVE (convex) on D. The following statements are equivalent, where x^* is an interior point of D:

(1) $\dfrac{\partial f(x^*)}{\partial x_i} = 0, \quad \text{for} \quad i = 1, 2, \ldots n$

(2) f achieves a [unique] GLOBAL MAXIMUM (global minimum) at x^*

Constrained Optimization

Equality Constraints

$$\max_{x_1, x_2} f(x_1, x_2) \qquad \text{s.t.} \qquad g(x_1, x_2) = 0$$

Constrained Optimization

Equality Constraints

$$\max_{x_1, x_2} f(x_1, x_2) \qquad \text{s.t.} \qquad g(x_1, x_2) = 0$$

Solve:

1. By substitution

Constrained Optimization

Equality Constraints

$$\max_{x_1, x_2} f(x_1, x_2) \qquad \text{s.t.} \qquad g(x_1, x_2) = 0$$

Solve:

1. By substitution
2. Lagrange’s Method

<u>**Constrained Optimization**</u>

Equality Constraints

<u>**Theorem**</u>**: Sufficient Conditions for Local Optima with Equality Constraints**

Let the objective function be $f(x)$ and the m constraints be $g^j(x) = 0, \quad j = 1, 2, \ldots m$

Let (x^*, Λ^*) solve the F.O.C. Then

1. x^* is a <u>**local**</u> **maximum** of $f(x)$ subject to the constraints, if the bordered principal minors, <u>evaluated at (x^*, Λ^*)</u>, **alternate in sign** beginning with negative

2. x^* is a <u>**local**</u> **minimum** of $f(x)$ subject to the constraints, if the bordered principal minors, <u>evaluated at (x^*, Λ^*)</u>, **are all negative**

Constrained Optimization

Inequality Constraints

Theorem: Necessary Conditions for Optima of Real-valued functions s.t. Nonnegative Constraints

Let the objective function $f(x)$ be continuously differentiable

1. If x^* **maximizes** $f(x)$ s.t. $x^* \geq 0$, then x^* satisfies:

(i) $\dfrac{\partial f(x)}{\partial x_i} \leq 0, \qquad i = 1, 2 \ldots n$

(ii) $x_i^* \left[\dfrac{\partial f(x^*)}{\partial x_i} \right] = 0, \qquad i = 1, 2 \ldots n$

(iii) $x_i^* \geq 0 \qquad i = 1, 2, \ldots n$

Constrained Optimization

Inequality Constraints

KUHN-TUCKER CONDITIONS

Value Functions

$$M(a) \equiv f(x(a), a)$$

<u>Theorem</u>: Theorem of the Maximum

If the objective function and the constraint are continuous in the parameters, and if the domain is a compact set, then $M(a)$ and $x(a)$ are **continuous** in a

THE ENVELOPE THEOREM

Consider the problem $\max_x f(x;a)$ s.t. $g(x;a)=0$ and $x \geq 0$

and suppose the objective function and constraint are continuously differentiable in a. For each a, let $x(a) >> 0$ **uniquely** solve the problem and assume that it is also continuously differentiable in the parameters a.

Let $L(x,a,\lambda)$ be the problem's associated Lagrangian function and let $(x(a),\lambda(a))$ solve the Kuhn-Tucker conditions. Finally, let $M(a)$ be the problem's associated maximum-value function.

Then the **Envelope Theorem** states that

$$\frac{\partial M(a)}{\partial a_j} = \left.\frac{\partial L}{\partial a_j}\right|_{\substack{x(a)\\ \lambda(a)}} \qquad j=1,2,\ldots m$$